hep-ph/9812268 SINP/TNP/98-34

Photogravitational processes

José F. Nieves
Department of Physics, University of Puerto Rico
Río Piedras, Puerto Rico 00939

Palash B. Pal
Saha Institute of Nuclear Physics, 1/AF Bidhan-Nagar
Calcutta 700064, INDIA

December 1998

Abstract

In the linearized theory of gravity coupled to the electromagnetic field, we calculate the cross sections of two processes in which both the gravitational and electromagnetic interactions play a role. Those processes, in which a graviton is produced in the final state, are: (a) the photoproduction of graviton off electrons, and (b) the production of a photon and a graviton from e^+-e^- pair annihilation. The motivation and outlook for these calculations are discussed.

1 Introduction

In the framework of Quantum Field Theory, the various fundamental interactions are understood as due to the exchange of gauge bosons. In the case of gravity, at least for weak gravitational fields, the linearized version of Einstein's theory of general relativity can be used to construct a quantum theory involving the spin-2 graviton which mediates the gravitational interactions.

Historically however, the interactions have been discovered by observing processes involving fermions, and not the gauge bosons, in the outer states. The gauge particles act as virtual mediators in such processes. At some later stage, the processes where the mediating particles also appear in the initial or the final state have been discovered. That was the case of electromagnetism, for example, where the interaction between electrons had been known for a long time, but Compton scattering and the photoelectric effect were discovered in the twentieth century. Similarly, while the weak interaction between fermions was known since the beginning of this century, the processes involving real W and Z bosons have been observed only very recently.

It seems to be a natural extension to this, to inquire about processes involving gravitons in the initial and/or the final state. As examples of such processes, in this article we consider two of them, which involve one graviton and a photon, and to which refer as "photogravitational processes". The first one is the photoproduction of graviton off electrons and the other one is the production of a photon and a graviton from e^+e^- annihilation.

Besides the intrinsic interest that the study of these processes have, there are some practical motivations for this undertaking that we should mention.

A few years ago we showed that, in a medium that contains electrons but not the other charged leptons, such as normal matter, the electromagnetic interactions of neutrinos are not the same for all the neutrino flavors[2]. It was observed there that, in the presence of a static magnetic field, the effective electromagnetic interactions of the neutrinos produce an additional contribution to the neutrino index of refraction which modifies the condition for resonant oscillations in matter[3]. This effect is the basis for the explanation of the large birth velocities of pulsars in terms of the asymmetric emission of neutrinos from the cooling protostar, which is produced by the matter-enhanced neutrino oscillations biased by the supernova's magnetic field [4].

Recently[1] we have shown that, in analogy with the fact that in a matter background the electron neutrinos have different electromagnetic interactions than the muon or tau neutrinos, their gravitational interactions also

differ. Furthermore, motivated by the fact that the gravitational interactions of neutrinos may be important in some contexts in neutrino physics[5, 6], in Ref. [1] we determined the effective gravitational interactions of neutrinos in a matter background, by calculating the one-loop contribution to the neutrino stress-energy tensor, which is the gravitational analog of the electromagnetic current. Those calculations were based on the linearized theory of gravitational couplings of fermions, including the interaction terms with the W and Z gauge bosons, and an outline of their derivation was presented in that reference. In addition, special attention was given to show that the result for the matter-induced gravitational vertex of the neutrino is (gravitational) gauge invariant.

In an outgrowth and extension of the work of Ref. [1] which is currently underway, the electromagnetic interactions, in addition to the gravitational ones, play a role. In that case, the result of the calculation should be gravitational and electromagnetic gauge invariant. In order to understand some subtle issues that have to do with the graviton couplings to the photon and (charged) fermions, we found convenient and instructive to consider processes involving both the gravitational and the electromagnetic interactions, but isolated from the additional complications that arise in the calculations of effective interactions in a matter background. The photogravitational processes that we have considered here, besides their intrinsic interest, serve the stated purpose well.

We should mention also that the processes that we consider here are not related to the soft graviton processes that are considered in connection with the equivalence principle and the soft graviton theorem[8]. Those are processes in which the graviton is radiated by one of the external particles, but the process can occur also without the radiated graviton. In contrast, the processes that we consider cannot occur in the absence of graviton emission.

We have organized the presentation as follows. In Section 2 we outline the derivation of the various gravitational couplings that are required in the calculation. The calculation of the cross section for the photoproduction of gravitons is carried out in Section 3, where special attention is given to check that the couplings derived in Section 2 ensure a gauge invariant result. Similarly, in Section 4 we consider the process of e^+e^- pair annihilation, and finally Section 5 contains our concluding remarks.

2 Interactions involving gravitons

In this section, we derive the various couplings involving the graviton which will be relevant for us. As already indicated, all the results are derived in linearized theory of gravity, in which the metric tensor is written as

$$g_{\lambda\rho} = \eta_{\lambda\rho} + 2\kappa h_{\lambda\rho} \tag{2.1}$$

where $\eta_{\lambda\rho}$ is the flat space metric. We then expand the Lagrangian in the presence of gravity in powers of κ and keep only the first order terms. In this formulation, $h_{\lambda\rho}$ appears as the graviton field, which is a spin-2 quantum field coupled to the stress-energy tensor, whose interactions can be studied in the flat Minkowskian background. The Einstein-Hilbert action for pure gravity is given by

$$\mathcal{A} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-\mathsf{g}}\,R\,, \tag{2.2}$$

where R is the Ricci scalar, g is the determinant of the matrix $g_{\lambda\rho}$, and G is the Newton's constant. Using Eq. (2.1), we can verify that this gives the correct kinetic terms for the spin-2 field if we make the identification

$$\kappa = \sqrt{8\pi G}\,. \tag{2.3}$$

We now discuss the couplings of the graviton to fermions and the photon.

2.1 Fermion couplings

These couplings were already considered in Ref. [1], and we refer to that work for the details of the derivation. The final result is that the coupling of the graviton field $h_{\lambda\rho}$ with the electron field can be written as

$$\mathcal{L}^{(ee)}_h = -\kappa h^{\lambda\rho}(x)\widehat{T}^{(e)}_{\lambda\rho}(x)\,, \tag{2.4}$$

where the stress-energy tensor operator $\widehat{T}^{(e)}_{\lambda\rho}$ for the electron field is given by

$$\widehat{T}^{(e)}_{\lambda\rho}(x) = \left\{\frac{i}{4}\overline{\psi}(x)\left[\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu\right]\psi(x) + \text{h.c.}\right\} - \eta_{\lambda\rho}\mathcal{L}^{(e)}_0(x)\,. \tag{2.5}$$

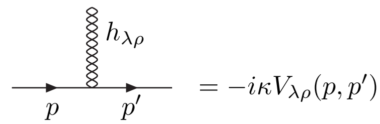

Figure 1: Feynman rule for the tree-level gravitational vertex of a fermion.

Here $\mathcal{L}_0^{(e)}(x)$ is the Lagrangian for the free Dirac field, which we write in the explicitly Hermitian form

$$\mathcal{L}_0^{(e)} = \left[\frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi + \text{h.c.}\right] - m\overline{\psi}\psi\,, \tag{2.6}$$

m being the mass of the electron. The Feynman rules derived from this interaction are presented in Fig. 1, where

$$V_{\lambda\rho}(p,p') = \frac{1}{4}\left[\gamma_\lambda(p+p')_\rho + \gamma_\rho(p+p')_\lambda\right] - \frac{1}{2}\eta_{\lambda\rho}\left[(\not{p} - m) + (\not{p}' - m)\right]\,. \tag{2.7}$$

For fermions on-shell these agree with the results quoted in standard textbooks[7]. However, the additional terms that appear in Eq. (2.7) for off-shell fermions are important for us, since there are virtual fermions in the diagrams that we have to consider.

2.2 Couplings involving photons

The graviton couplings that involve the photon field are determined by the Lagrangian density

$$\mathcal{L}_{\rm em} = -\frac{1}{4}\sqrt{-\mathsf{g}}\,F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} + e\sqrt{-\mathsf{g}}\,\overline{\psi}\gamma^a {v_a}^\mu\psi A_\mu\,, \tag{2.8}$$

where in our notation the electric charge of the electron is $-e$, and ${v_a}^\mu$ are the vierbeins (or tetrads) which satisfy the conditions

$$\eta^{ab}{v_a}^\mu v_b^\nu = g^{\mu\nu}\,, \qquad g_{\mu\nu}{v_a}^\mu {v_b}^\nu = \eta_{ab}\,. \tag{2.9}$$

These are necessary also for deriving the interaction in Eq. (2.4) by expanding in terms of κ, a procedure which has been detailed in Ref. [1]. Mimicking that procedure here we put

$${v_a}^\mu \quad \simeq \quad {\eta_a}^\mu - \kappa {h_a}^\mu\,, \tag{2.10}$$

which follows from Eqs. (2.1) and (2.9), and also

$$\sqrt{-\mathsf{g}} = 1 + \kappa\eta_{\mu\nu}h^{\mu\nu} \tag{2.11}$$

to first order in κ.

Substituting these in Eq. (2.8), we obtain the interaction terms involving the graviton,

$$\mathcal{L}_h^{(\rm em)} = \mathcal{L}_h^{(eA)} + \mathcal{L}_h^{(AA)}\,, \tag{2.12}$$

where

$$\mathcal{L}_h^{(eA)} = e\kappa h^{\lambda\rho}a_{\mu\nu\lambda\rho}\overline{\psi}\gamma^\nu\psi A^\mu\,, \tag{2.13}$$

with

$$a_{\mu\nu\lambda\rho} = \eta_{\mu\nu}\eta_{\lambda\rho} - \frac{1}{2}\left(\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\nu\lambda}\eta_{\mu\rho}\right)\,. \tag{2.14}$$

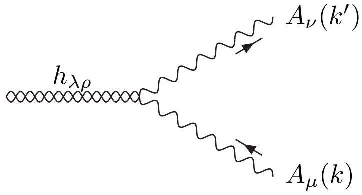

Figure 2: Feynman rule for the coupling of the photon with the graviton.

Notice that, in order to obtain the correct field equations from Eq. (2.8), it is important that $g^{\mu\nu}$ (or equivalently ${v_a}^{\mu}$) and A_μ are treated as the independent variables and not, for example, the contravariant vector A^μ. If we were to use A^μ as the independent variable, the couplings obtained would not be the correct ones and the resulting amplitudes would not be gauge invariant.[1] However, once we separate out the first order terms in κ, we can raise and lower the indices in those terms by the flat metric $\eta_{\mu\nu}$ since we are interested in the first order terms in the final result. Thus, in those terms, we can blatantly use upper or lower indices, as we have done in Eq. (2.13).

The photon couplings with the graviton are determined as

$$\mathcal{L}_h^{(AA)} = -\kappa h^{\lambda\rho}(x)\widehat{T}^{(A)}_{\lambda\rho}(x)\,, \tag{2.15}$$

where

$$\widehat{T}^{(A)}_{\lambda\rho} \quad = \quad \frac{1}{4}\eta_{\lambda\rho}F_{\mu\nu}F^{\mu\nu} + F_{\lambda\mu}{F^{\mu}}_{\rho} \tag{2.16}$$

is the stress-energy tensor operator for the photon field. The Feynman rule for the photon-photon-graviton vertex is parametrized in Fig. 2. With the momentum convention shown in Fig. 2, $C_{\mu\nu\lambda\rho}$ is given by

$$\begin{aligned} C_{\mu\nu\lambda\rho}(k,k') \quad = \quad & \eta_{\lambda\rho}(\eta_{\mu\nu}k\cdot k' - k'_\mu k_\nu) - \eta_{\mu\nu}(k_\lambda k'_\rho + k'_\lambda k_\rho) \\ & + k_\nu(\eta_{\lambda\mu}k'_\rho + \eta_{\rho\mu}k'_\lambda) + k'_\mu(\eta_{\lambda\nu}k_\rho + \eta_{\rho\nu}k_\lambda) \\ & - k\cdot k'(\eta_{\lambda\mu}\eta_{\rho\nu} + \eta_{\lambda\nu}\eta_{\rho\mu}) \end{aligned} \tag{2.17}$$

This is the same as what is given in Scadron's book, except for an overall factor of 2 which we believe is a mistake or misprint in the book.

3 Photoproduction of gravitons

3.1 The amplitude and its invariances

In this section we consider the process

$$e^-(p) + \gamma(k) \to e^-(p') + \mathcal{G}(q)\,, \tag{3.1}$$

for which the tree level diagrams are shown in Fig. 3. Writing the Feynman amplitude as

$$\begin{aligned} iM \quad &= \quad (-i\kappa)(ie)i\,[\overline{u}(p')\Gamma_{\mu\lambda\rho}u(p)]\,\varepsilon^{*\lambda\rho}(q)\epsilon^\mu(k) \\ &\equiv \quad ie\kappa\mathcal{M}\,, \end{aligned} \tag{3.2}$$

the contributions to $\Gamma_{\mu\lambda\rho}$ from the various diagrams are given by

$$\Gamma^{(a)}_{\mu\lambda\rho} \quad = \quad V_{\lambda\rho}(p'+q,p')S_F(p'+q)\gamma_\mu \tag{3.3}$$

$$\Gamma^{(b)}_{\mu\lambda\rho} \quad = \quad \gamma_\mu S_F(p-q)V_{\lambda\rho}(p,p-q) \tag{3.4}$$

$$\Gamma^{(c)}_{\mu\lambda\rho} \quad = \quad \gamma^\alpha a_{\alpha\mu\lambda\rho} \tag{3.5}$$

$$\begin{aligned} \Gamma^{(d)}_{\mu\lambda\rho} \quad &= \quad \gamma_\alpha D_F^{\alpha\nu}(k-q)C_{\mu\nu\lambda\rho}(k,k-q) \\ &= \quad -\gamma^\nu\Delta(k-q)C_{\mu\nu\lambda\rho}(k,k-q)\,, \end{aligned} \tag{3.6}$$

[1]Such a wrong coupling was in fact presented in Ref. [1]. However, it turns out that the final results derived in that paper are still correct, although the amplitudes corresponding to the individual diagrams are not. This will be discussed in detail elsewhere.

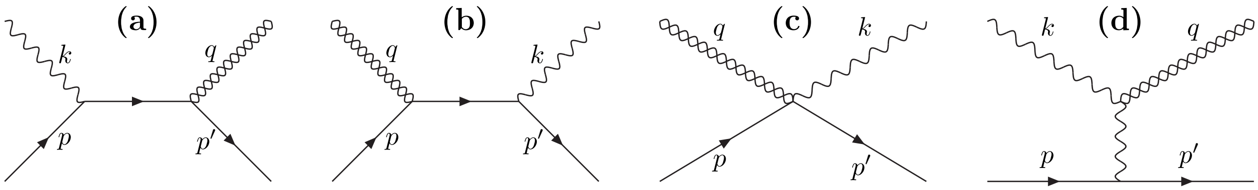

Figure 3: Tree level diagrams for the photoproduction of gravitons off electrons. k and q label the lines of the incomming photon and outgoing graviton, respectively.

where $a_{\mu\nu\lambda\rho}$ and $C_{\mu\nu\lambda\rho}$ have been defined in Eqs. (2.14) and (2.17), and we have written the photon propagator as

$$D_F^{\mu\nu}(k') = -\eta^{\mu\nu}\Delta(k') = -\frac{\eta^{\mu\nu}}{k'^2}\,. \tag{3.7}$$

As usual, the spinors that appear in Eq. (3.2) satisfy

$$\begin{aligned} \overline{u}(p')\not{p}' &= m\overline{u}(p') & (3.8)\\ \not{p}u(p) &= mu(p)\,. & (3.9) \end{aligned}$$

3.1.1 Checking gravitational gauge invariance

Gravitational gauge invariance amounts to the statement that the condition

$$q^\lambda \overline{u}(p')\Gamma_{\mu\lambda\rho}u(p) = 0 \tag{3.10}$$

holds when all particles, other than the graviton, are on-shell. In order to verify this, let us consider first $\Gamma^{(a)}_{\mu\lambda\rho}$. Note that

$$q^\lambda V_{\lambda\rho}(p'+q,p') = \frac{1}{4}[\not{q}(2p'+q)_\rho + \gamma_\rho(2p'+q)\cdot q] - \frac{1}{2}\not{q}q_\rho\,, \tag{3.11}$$

where we have used Eq. (3.8). We now use the identities

$$(2p'\cdot q + q^2)\; S_F(p'+q) = \not{p}' + \not{q} + m\,, \tag{3.12}$$

and

$$\begin{aligned} \not{q} &= S_F^{-1}(p'+q) - S_F^{-1}(p') \\ &\rightarrow S_F^{-1}(p'+q)\,, & (3.13) \end{aligned}$$

where the last step follows because of Eq. (3.8). This finally gives

$$q^\lambda\Gamma^{(a)}_{\mu\lambda\rho} = \frac{1}{4}[4p'_\rho - q_\rho + \gamma_\rho\not{q}]\gamma_\mu\,. \tag{3.14}$$

For the diagram (b) the procedure is similar and the result for that case is

$$q^\lambda\Gamma^{(b)}_{\mu\lambda\rho} = \frac{1}{4}\gamma_\mu[-4p_\rho - q_\rho + \not{q}\gamma_\rho]\,. \tag{3.15}$$

Also, trivially,

$$q^\lambda\Gamma^{(c)}_{\mu\lambda\rho} = \gamma_\mu q_\rho - \frac{1}{2}\gamma_\rho q_\mu - \frac{1}{2}\eta_{\mu\rho}\not{q}\,. \tag{3.16}$$

Finally, for the diagram (d), we need to use also the on-shell conditions for the photon

$$\begin{aligned} \epsilon^\mu(k)k_\mu &= 0\,, \\ k^2 &= 0\,, \end{aligned} \tag{3.17}$$

which imply

$$\gamma_\nu q^\lambda C_{\mu\nu\lambda\rho}(k,k-q) = [\gamma_\mu k_\rho - \eta_{\mu\rho}\not{k}]\,\Delta^{-1}(k-q) + (\not{k}-\not{q})\,[q_\mu k_\rho - \eta_{\mu\rho} k\cdot q]\;. \tag{3.18}$$

The last term vanishes between the spinors since $\not{k}-\not{q} = \not{p}' - \not{p}$, so we obtain

$$q^\lambda \Gamma^{(d)}_{\mu\lambda\rho} = -\gamma_\mu k_\rho + \eta_{\mu\rho}\not{k}\,. \tag{3.19}$$

Adding Eqs. (3.14) - (3.16) and (3.19), and using

$$\frac{1}{4}[\gamma_\rho \not{q}\gamma_\mu + \gamma_\mu \not{q}\gamma_\rho] = \frac{1}{2}[\gamma_\mu q_\rho + \gamma_\rho q_\mu - \eta_{\mu\rho}\not{q}]\,, \tag{3.20}$$

we finally obtain

$$q^\lambda \Gamma_{\mu\lambda\rho} = \gamma_\mu(p'-p-k+q)_\rho + \eta_{\mu\rho}(\not{k}-\not{q})\,, \tag{3.21}$$

which leads to Eq. (3.10) because the first term is zero due to momentum conservation and the second vanishes when it is multiplied by the spinors.

3.1.2 Checking electromagnetic gauge invariance

The amplitude should also be invariant under electromagnetic gauge transformations. To check this, we note that

$$k^\mu \Gamma^{(a)}_{\mu\lambda\rho} = V_{\lambda\rho}(p'+q,p')S_F(p+k)\not{k}\,, \tag{3.22}$$

using the conservation of momentum to write the fermion propagator in terms of $p+k$ rather than $p'+q$ here. Writing $\not{k} = (\not{p}+\not{k}-m) - (\not{p}-m)$ and noting that $(\not{p}-m)$ vanishes when it acts on the spinor on the right, we obtain $S_F(p+k)\not{k} = 1$, so that

$$k^\mu \Gamma^{(a)}_{\mu\lambda\rho} = V_{\lambda\rho}(p'+q,p')\,. \tag{3.23}$$

Similarly,

$$k^\mu \Gamma^{(b)}_{\mu\lambda\rho} = -V_{\lambda\rho}(p,p-q)\,, \tag{3.24}$$

and, trivially,

$$k^\mu \Gamma^{(c)}_{\mu\lambda\rho} = \eta_{\lambda\rho}\not{k} - \frac{1}{2}(\gamma_\rho k_\lambda + \gamma_\lambda k_\rho) \tag{3.25}$$

$$k^\mu \Gamma^{(d)}_{\mu\lambda\rho} = 0\,. \tag{3.26}$$

Adding Eqs. (3.23) - (3.26) and using the explicit forms of $V_{\lambda\rho}$ we obtain

$$k^\mu \Gamma_{\mu\lambda\rho} = \eta_{\lambda\rho}(\not{k}-\not{q})\,, \tag{3.27}$$

which vanishes between the spinors. Thus, the amplitude satisfies electromagnetic gauge invariance as well.

It is interesting to note in this context that, in order to verify the electromagnetic gauge invariance, there is no need to assume that the graviton is on-shell, whereas we had to put the photon on-shell to prove gravitational gauge invariance. The reason for this is the following. In the proof of a given kind of gauge invariance, only the particles carrying the corresponding charge need to be on-shell. The graviton is electromagnetically neutral so that only the electrons have to be put on-shell in proving electromagnetic gauge invariance. On the other hand, the electrons as well as the photons have gravitational couplings; i.e., both are "gravitationally charged". So, the amplitude is transverse with respect to the graviton only if the electrons and the photon are all on-shell.

3.2 Calculation of the cross section

We will calculate the cross section of the process in the lab frame, in which the initial electron is at rest, and therefore we set

$$p^\mu = (m, 0, 0, 0)\,. \tag{3.28}$$

In that frame we also write

$$\begin{aligned} k^\mu &= (\omega, \boldsymbol{k}) \\ q^\mu &= (\omega', \boldsymbol{q})\,. \end{aligned} \tag{3.29}$$

The polarization vector ϵ^μ of the photon satisfies

$$\epsilon^\mu \epsilon_\mu = -1\,, \tag{3.30}$$

in addition to the transversality condition given in Eq. (3.17). Since the amplitude is gauge invariant, we can use the freedom of adding any multiple of k^μ to the polarization vector to ensure that its time component is zero, i.e.,

$$\epsilon^\mu = (0, \boldsymbol{\epsilon})\,, \tag{3.31}$$

which defines the *radiation gauge*. In the present context, this choice of gauge ensures that

$$p \cdot \epsilon = 0\,, \tag{3.32}$$

which will be very useful to simplify the expression for the amplitude encountered below.

Similarly, the polarization tensor $\varepsilon^{\lambda\rho}$ for the graviton is a symmetric tensor which satisfies the conditions

$$\varepsilon^{\lambda\rho}\varepsilon_{\lambda\rho} = 1\,, \tag{3.33}$$

$$\varepsilon^{\lambda\rho}(q)q_\lambda = \varepsilon^{\lambda\rho}(q)q_\rho = 0\,, \tag{3.34}$$

$$\varepsilon^{\lambda\rho}\eta_{\lambda\rho} = 0\,. \tag{3.35}$$

In addition, we can choose the 'radiation gauge' defined by

$$\varepsilon^{0\rho} = \varepsilon^{\lambda 0} = 0\,, \tag{3.36}$$

so that

$$p_\lambda \varepsilon^{\lambda\rho} = p_\rho \varepsilon^{\lambda\rho} = 0\,. \tag{3.37}$$

With these definitions, we can now write down the simplified form for the various terms in $\mathcal{M}$ as defined in Eq. (3.2), which we label with the same letter that labels the corresponding contribution to $\Gamma_{\mu\lambda\rho}$.

Since the $\eta_{\lambda\rho}$ terms do not contribute to the physical amplitude due to Eq. (3.35), we can write

$$\begin{aligned} \mathcal{M}_a &= \frac{1}{4}\overline{u}(p')\left[(2p'+q)_\lambda\gamma_\rho + (2p'+q)_\rho\gamma_\lambda\right]\frac{1}{\not{p}+\not{k}-m}\gamma_\mu u(p)\varepsilon^{*\lambda\rho}\epsilon^\mu \\ &= \overline{u}(p')p'_\lambda \not{\varepsilon}^{*\lambda}\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\not{\epsilon}u(p)\,, \end{aligned} \tag{3.38}$$

where we have used Eq. (3.34) and the shorthand notation

$$\not{\varepsilon}^{*\lambda} \equiv \gamma_\rho \varepsilon^{*\lambda\rho} = \gamma_\rho \varepsilon^{*\rho\lambda}\,. \tag{3.39}$$

On the other hand,

$$(\not{p}+m)\not{\epsilon}u(p) = [2p\cdot\epsilon + \not{\epsilon}(-\not{p}+m)]u(p) = 0\,, \tag{3.40}$$

where the last equality follows from (3.32) and the Dirac equation for the spinor. Using $(p+k)^2 - m^2 = 2m\omega$, it then follows that

$$\mathcal{M}_a = \frac{k_\lambda}{2m\omega}\overline{u}(p')\not{\varepsilon}^{*\lambda}\not{k}\not{\epsilon}u(p)\,, \tag{3.41}$$

where we have used momentum conservation and Eqs. (3.34) and (3.37) to replace p'_λ by k_λ.

Regarding $\mathcal{M}_b$, we find that

$$\mathcal{M}_b \quad = \quad 0\,, \tag{3.42}$$

since either a p or a q contracts with the graviton polarization tensor, while

$$\mathcal{M}_c \quad = \quad -\,[\overline{u}(p')\not\epsilon^{*\lambda}u(p)]\epsilon_\lambda\,. \tag{3.43}$$

Finally, for $\mathcal{M}_d$, we note that $C_{\mu\nu\lambda\rho}$, defined in Eq. (2.17), can be written as

$$C_{\mu\nu\lambda\rho}(k,k-q) = 2\left(-\eta_{\mu\nu}k_\lambda k_\rho + \eta_{\lambda\mu}k_\nu k_\rho - \eta_{\lambda\nu}q_\mu k_\rho + \eta_{\lambda\mu}\eta_{\rho\nu}k\cdot q\right) + \cdots\,, \tag{3.44}$$

where the dots indicate terms which vanish after contraction with the polarization factors, either because they are antisymmetric in the indices λ and ρ, or because of various on-shell conditions described above. Therefore,

$$\mathcal{M}_d \quad = \quad \frac{1}{k\cdot q}\overline{u}(p')\gamma^\nu u(p)\left(-\eta_{\mu\nu}k_\lambda k_\rho + \eta_{\lambda\mu}k_\nu k_\rho - \eta_{\lambda\nu}q_\mu k_\rho + \eta_{\lambda\mu}\eta_{\rho\nu}k\cdot q\right)\varepsilon^{*\lambda\rho}\epsilon^\mu\,. \tag{3.45}$$

If we add $\mathcal{M}_c$ and $\mathcal{M}_d$ the last term in Eq. (3.45) cancels $\mathcal{M}_c$, so that

$$\mathcal{M}'_d \equiv \mathcal{M}_c + \mathcal{M}_d \quad = \quad \frac{1}{k\cdot q}\overline{u}(p')\gamma^\mu u(p)\left[-\epsilon_\mu k\cdot\zeta^* + k_\mu \epsilon\cdot\zeta^* - q\cdot\epsilon\zeta^*_\mu\right]\,, \tag{3.46}$$

where we have defined

$$\zeta^\mu \equiv \varepsilon^{\mu\nu}k_\nu\,. \tag{3.47}$$

We now square the amplitude and sum over the final electron spin and average over the initial spin. However, we do not sum or average over the polarizations of the photon nor of the graviton. This gives, for the different terms,

$$\frac{1}{2}\sum_{\text{spin}}\mathcal{M}^*_a\mathcal{M}_a \quad = \quad \frac{1}{m\omega}\left[2|k\cdot\zeta|^2 - m\omega'(\zeta\cdot\zeta^*)\right]\,, \tag{3.48}$$

$$\frac{1}{2}\sum_{\text{spin}}\mathcal{M}'^*_d\mathcal{M}_d \quad = \quad \frac{2}{(k\cdot q)^2}\left[2m^2\omega\omega'|\epsilon\cdot\zeta^*|^2 + m(\omega-\omega')\left\{|k\cdot\zeta|^2 - (q\cdot\epsilon)^2(\zeta\cdot\zeta^*)\right\}\right] \tag{3.49}$$

$$\frac{1}{2}\sum_{\text{spin}}\mathcal{M}^*_a\mathcal{M}'_d \quad = \quad \frac{1}{m\omega k\cdot q}\left[m\omega(q\cdot\epsilon)^2(\zeta\cdot\zeta^*) - 2m^2\omega\omega'|\epsilon\cdot\zeta^*|^2 - m(2\omega-\omega')|k\cdot\zeta|^2\right] \tag{3.50}$$

$$\frac{1}{2}\sum_{\text{spin}}\mathcal{M}'^*_d\mathcal{M}_a \quad = \quad \frac{1}{2}\sum_{\text{spin}}\mathcal{M}^*_a\mathcal{M}'_d\,, \tag{3.51}$$

where we have taken ϵ^μ to be real, which corresponds to a linearly polarized photon. Adding these terms and using the relation $k\cdot q = m(\omega-\omega')$, we obtain

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{2}\sum_{\text{spin}}\mathcal{M}^*\mathcal{M} \quad = \quad \frac{4\omega'^2}{(\omega-\omega')^2}|\epsilon\cdot\zeta^*|^2 - \frac{\omega'}{\omega}(\zeta\cdot\zeta^*)\,. \tag{3.52}$$

Using the conditions given in Eqs. (3.31) and (3.36), it can be written as

$$\overline{|\mathcal{M}|^2} \quad = \quad \frac{4\omega'^2}{(\omega-\omega')^2}|\boldsymbol{\epsilon}\cdot\boldsymbol{\zeta}^*|^2 + \frac{\omega'}{\omega}|\boldsymbol{\zeta}|^2\,, \tag{3.53}$$

where, according to Eq. (3.47)

$$\zeta^i = \varepsilon^{ij}k_j\,. \tag{3.54}$$

In order to obtain the differential cross section for an unpolarized beam of photons, we must to sum over photon polarizations and divide by 2. The sum over the polarizations is carried out by using

$$\sum \epsilon^*_i(k)\epsilon_j(k) = \delta_{ij} - \widehat{k}_i\widehat{k}_j\,, \tag{3.55}$$

where $\widehat{k}$ denotes the unit vector in the direction of $\boldsymbol{k}$. In this way, the formula that we obtain for the squared amplitude, summing over the final electron spin and averaging over the initial electron and photon polarizations is

$$\overline{|\mathcal{M}|^2} \;=\; \frac{\omega'}{\omega}|\boldsymbol{\zeta}|^2 + \frac{2\omega'^{\,2}}{(\omega-\omega')^2}\left(|\boldsymbol{\zeta}|^2 - \frac{|\boldsymbol{k}\cdot\boldsymbol{\zeta}|^2}{\omega^2}\right). \tag{3.56}$$

Further, we can represent the graviton polarization tensor by

$$\varepsilon^{\mu\nu}(q) = \epsilon^{\nu}(\boldsymbol{q})\epsilon^{\nu}(\boldsymbol{q})\,, \tag{3.57}$$

where

$$\epsilon^{\mu}(\boldsymbol{q}) = (0, \boldsymbol{\epsilon}(\boldsymbol{q})) \tag{3.58}$$

are the spin-1 polarization vectors for a definite helicity $(\pm)$. Then it follows that

$$\begin{aligned} |\boldsymbol{k}\cdot\boldsymbol{\zeta}|^2 &= \omega^4|\widehat{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{q})|^4 \\ |\boldsymbol{\zeta}|^2 &= \omega^2|\widehat{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{q})|^2\,, \end{aligned} \tag{3.59}$$

so that

$$\overline{|\mathcal{M}|^2} \;=\; \omega'\omega|\widehat{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{q})|^2 + \frac{2\omega'^{\,2}\omega^2}{(\omega-\omega')^2}\left(|\widehat{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{q})|^2 - |\widehat{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{q})|^4\right). \tag{3.60}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha G}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\overline{|\mathcal{M}|^2} \tag{3.61}$$

where

$$\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}\,, \tag{3.62}$$

θ being the angle between $\boldsymbol{q}$ and $\boldsymbol{k}$.

Let us write the components of $\widehat{q}$ in the form

$$\widehat{q} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3.63}$$

in a system of axis where $\boldsymbol{k}$ is along the z-direction. Then

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2) \tag{3.64}$$

where

$$\begin{aligned} \boldsymbol{\epsilon}_1 &\equiv (-\sin\phi, \cos\phi, 0) \\ \boldsymbol{\epsilon}_2 &\equiv (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\,. \end{aligned} \tag{3.65}$$

From these relations we then have

$$|\widehat{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{q})|^2 = \frac{1}{2}\sin^2\theta\,, \tag{3.66}$$

and finally

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\left[\omega\omega'\left(1-\cos^2\theta\right) + \frac{\omega^2\omega'^{\,2}}{(\omega-\omega')^2}\left(1-\cos^4\theta\right)\right], \tag{3.67}$$

for either helicity of the graviton. Summing over the graviton helicities, the total differential cross section is then given by

$$\frac{d\sigma_{tot}}{d(\cos\theta)} = \frac{\pi\alpha G}{m^2}\omega'^{\,2}\left[\frac{\omega'}{\omega}\left(1-\cos^2\theta\right) + \frac{\omega'^{\,2}}{(\omega-\omega')^2}\left(1-\cos^4\theta\right)\right], \tag{3.68}$$

where ω' depends on $\cos\theta$ through Eq. (3.62). The second term in square brackets in Eq. (3.68) exhibits the singularity in the forward direction that is typical of the Coulomb potential, which is due to its long range character. In the present case, it has made its way through the photon exchange diagram of Fig. 3.

4 Gravity from pair annihilation

We now consider the related process

$$e^{-}(p) + e^{+}(p') \to \gamma(k) + \mathcal{G}(q)\,, \tag{4.1}$$

where a graviton and a photon are produced due to the annihilation of an electron-positron pair. The amplitude for this process is obtained from Eq. (3.2) by making the substitutions

$$\begin{aligned} p' &\to -p'\,, \qquad k \to -k & (4.2)\\ u(p') &\to v(p')\,, \qquad \epsilon^{\mu} \to \epsilon^{*\mu}\,. & (4.3) \end{aligned}$$

If we write the amplitude in the form

$$\begin{aligned} iM &= (-i\kappa)(ie)i\left[\overline{v}(p')\Gamma_{\mu\lambda\rho}u(p)\right]\varepsilon^{*\lambda\rho}(q)\epsilon^{*\mu}(k)\\ &\equiv ie\kappa\mathcal{M}\,, \end{aligned} \tag{4.4}$$

the contributions to $\Gamma_{\mu\lambda\rho}$ from the various diagrams are

$$\begin{aligned} \Gamma^{(a)}_{\mu\lambda\rho} &= V_{\lambda\rho}(q-p',-p')S_F(q-p')\gamma_\mu & (4.5)\\ \Gamma^{(b)}_{\mu\lambda\rho} &= \gamma_\mu S_F(p-q)V_{\lambda\rho}(p,p-q) & (4.6)\\ \Gamma^{(c)}_{\mu\lambda\rho} &= \gamma^\alpha a_{\alpha\mu\lambda\rho} & (4.7)\\ \Gamma^{(d)}_{\mu\lambda\rho} &= -\gamma^\nu\Delta(-k-q)C_{\mu\nu\lambda\rho}(-k,-k-q)\,, & (4.8) \end{aligned}$$

with the notation for vertices and propagators used in the previous sections. The explicit verification of gauge invariance, both gravitational and electromagnetic, is similar to the exercise of Sec. 3.

The on-shell relations

$$\begin{aligned} k^2 = q^2 &= 0\\ p^2 = p'^2 &= m^2\,, \end{aligned} \tag{4.9}$$

together with Eq. (3.35), allow us to write the corresponding contributions to $\mathcal{M}$ as

$$\begin{aligned} \mathcal{M}_a &= \frac{p'_\lambda}{2p\cdot k}\left[\overline{v}(p')\gamma_\rho(\not{q} - \not{p}' + m)\gamma_\mu u(p)\right]\varepsilon^{*\lambda\rho}\epsilon^{*\mu}\,, & (4.10)\\ \mathcal{M}_b &= -\frac{p_\lambda}{2p'\cdot k}\left[\overline{v}(p')\gamma_\mu(\not{p} - \not{q} + m)\gamma_\rho u(p)\right]\varepsilon^{*\lambda\rho}\epsilon^{*\mu}\,, & (4.11)\\ \mathcal{M}'_d \equiv \mathcal{M}_c + \mathcal{M}_d &= \frac{k_\lambda}{k\cdot q}\left[\overline{v}(p')\gamma^\nu u(p)\right](\eta_{\mu\nu}k_\rho - \eta_{\mu\rho}k_\nu - \eta_{\nu\rho}q_\mu)\,\varepsilon^{*\lambda\rho}\epsilon^{*\mu}\,. & (4.12) \end{aligned}$$

We will calculate the cross section in the center of mass frame of the initial particles. The following considerations then simplify the expression for the amplitude. Since

$$p + p' = (\sqrt{s}, \mathbf{0}) \tag{4.13}$$

in that frame, then in the radiation gauge for the graviton it follows that

$$k_\lambda \varepsilon^{\lambda\rho} = 0 \tag{4.14}$$

and

$$p'_\lambda \varepsilon^{\lambda\rho} = -p_\lambda \varepsilon^{\lambda\rho}\,, \tag{4.15}$$

with analogous identities for the contractions with respect to the index ρ. Eq. (4.14) tells us that $\mathcal{M}'_d = 0$. The expressions for $\mathcal{M}_{a,b}$ can be reduced further with the help of the relations

$$\begin{aligned} \not{p}u &= mu\\ \overline{v}\not{p}' &= -m\overline{v}\,. \end{aligned} \tag{4.16}$$

In this way we finally obtain

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2\,, \tag{4.17}$$

with

$$\begin{aligned} \mathcal{M}_1 &= \left[\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right](\overline{v}\not{\xi}^* u) \\ \mathcal{M}_2 &= \frac{1}{2p\cdot k}\,(\overline{v}\not{\xi}^*\not{k}\not{\epsilon}^* u) - \frac{1}{2p'\cdot k}\,(\overline{v}\not{\epsilon}^*\not{k}\not{\xi}^* u)\,. \end{aligned} \tag{4.18}$$

where we have defined the vector

$$\xi^\mu \equiv \varepsilon^{\mu\nu} p_\nu\,. \tag{4.19}$$

Notice that while we have chosen a specific gauge for the graviton polarization tensor in order to arrive at Eq. (4.18), the gauge for the photon polarization vector has been left unspecified. In particular, the expressions given in Eq. (4.18) become zero if the replacement $\epsilon_\mu \to k_\mu$ is made. Therefore, in the square of the amplitudes, the sum over the polarizations of the photon can be carried out by means of the formula

$$\sum \epsilon^\mu \epsilon^{*\nu} = -\eta^{\mu\nu}\,. \tag{4.20}$$

Thus, taking the square of the amplitude in Eq. (4.17), and averaging over the initial electron and positron spins and summing over the photon polarizations, we obtain

$$\begin{aligned} \overline{|\mathcal{M}_1|}^2 &= \left(\frac{m^2}{(p'\cdot k)^2} + \frac{m^2}{(p\cdot k)^2} - \frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)}\right)\left[2(p\cdot\xi)(p\cdot\xi^*) + p\cdot p'(\xi\cdot\xi^*) + m^2(\xi\cdot\xi^*)\right] \\ \overline{|\mathcal{M}_2|}^2 &= -\left(\frac{p\cdot k}{p'\cdot k} + \frac{p'\cdot k}{p\cdot k}\right)(\xi\cdot\xi^*) \\ 2\overline{\mathcal{M}_1\mathcal{M}_2^*} &= -\left(\frac{p'\cdot k}{(p\cdot k)^2} + \frac{p\cdot k}{(p'\cdot k)^2}\right)m^2|\xi|^2 \\ &\quad + \left(\frac{1}{p\cdot k} + \frac{1}{p'\cdot k}\right)\left[-m^2(\xi\cdot\xi^*) + 2p\cdot p'(\xi\cdot\xi^*) + 4(p\cdot\xi)(p\cdot\xi^*)\right]\,. \end{aligned} \tag{4.21}$$

Adding these, and using the relation

$$p\cdot p' + m^2 = (p+p')\cdot k\,, \tag{4.22}$$

the total amplitude squared is then given by

$$\overline{|\mathcal{M}|^2} = 2m^2\left(\frac{1}{p\cdot k} + \frac{1}{p'\cdot k}\right)^2 (p\cdot\xi)(p\cdot\xi^*) - \left(\frac{1}{p\cdot k} + \frac{1}{p'\cdot k}\right)(\xi\cdot\xi^*)\,. \tag{4.23}$$

To proceed further, we make a construction of the graviton polarization tensor in a way analogous to that given in Eq. (3.57). Thus, introducing the angle θ by writing

$$\cos\theta \equiv \widehat{q}\cdot\widehat{p}\,, \tag{4.24}$$

we have

$$\begin{aligned} (p\cdot\xi)(p\cdot\xi^*) &= \frac{1}{4}|\boldsymbol{p}|^4\sin^4\theta \\ \xi\cdot\xi^* &= -\frac{1}{2}|\boldsymbol{p}|^2\sin^2\theta\,, \end{aligned} \tag{4.25}$$

where $|\boldsymbol{p}|$ is the magnitude of the electron momentum, given by

$$|\boldsymbol{p}| = \frac{1}{2}\beta\sqrt{s}\,, \tag{4.26}$$

with

$$\beta = \sqrt{1 - \frac{4m^2}{s}} \tag{4.27}$$

being the electron velocity. In addition, we use the kinematic formulas

$$\begin{aligned} p \cdot k &= \frac{1}{4} s(1 + \beta\cos\theta) \\ p' \cdot k &= \frac{1}{4} s(1 - \beta\cos\theta) \,. \end{aligned} \tag{4.28}$$

Substituting these in Eq. (4.23) we finally arrive at

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{2} m^2 \beta^4 \sin^4\theta \left[\frac{1}{1+\beta\cos\theta} + \frac{1}{1-\beta\cos\theta}\right]^2 \\ &\quad + \frac{1}{8} s\beta^2 \sin^2\theta \left[\frac{1-\beta\cos\theta}{1+\beta\cos\theta} + \frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right] \,, \end{aligned} \tag{4.29}$$

which holds for either helicity of the graviton. If we sum over the graviton helicities, we must multiply Eq. (4.29) by a factor of 2. The total differential cross section, summed over the graviton helicities, is then given by

$$\begin{aligned} \frac{d\sigma_{\rm tot}}{d(\cos\theta)} &= \frac{\pi G \alpha \beta}{s} \left\{ m^2 \beta^2 \sin^4\theta \left[\frac{1}{1+\beta\cos\theta} + \frac{1}{1-\beta\cos\theta}\right]^2 \right. \\ &\quad \left. + \frac{s}{4} \sin^2\theta \left[\frac{1-\beta\cos\theta}{1+\beta\cos\theta} + \frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right] \right\} \,. \end{aligned} \tag{4.30}$$

It is curious to observe that, at energies of the order of $\sqrt{s} \simeq 10^{17}$ GeV, such as those that may have been available in the Early Universe, $\sigma_{\rm tot}$ becomes of the order of a few percent of a typical cross section $\sigma_0 \simeq (\alpha^2/s)$ for producing photons and the weak gauge bosons in analogus reactions. Of course, at such energies the non-linear terms in the gravitational interactions might play a significant role and change the cross section appreciably.

5 Conclusions

Unless our experience and current understanding has totally misled us, it is likely that, at some level, the graviton participates actively in physical processes as a real particle and not just as a mediator of the gravitational interactions.

In this work we have considered in some detail the calculation of the cross section for two such processes, in which a graviton emerges in the final state, and which involve both the gravitational and electromagnetic interactions. Special attention has been put in the derivation of the appropriate couplings that should be used in the Lagrangian, and in showing that with the proper choice of them the amplitudes for the processes are gauge invariant.

Apart from the intrinsic interest and from any direct application that these calculations might have, they provide a useful setting to study some of the subtle technical issues that are involved in this type of calculation, disentangled from the additional complications that appear in the study of processes that occur in a background medium. As already remarked in the Introduction and commented upon in Section 2, the details of the present work are relevant for the kind of calculation carried out in Ref. [1], which have to do with the matter-induced gravitational interactions of neutrinos and other particles in a background medium. These calculations, which extend the work of Ref. [1], are currently in progress.

References

[1] J. F. Nieves and P. B. Pal, Phys.Rev. D**58**, 096005 (1998).

[2] J. C. D'Olivo, J. F. Nieves and P. B. Pal, Phys. Rev. D**40**, 3679 (1989).

[3] See also, S. Esposito and G. Capone, Z. Phys. **C70**, 55 (1996);
P. Elmfors, D. Grasso and G. Raffelt, Nucl. Phys. B**479**, 3 (1996);
A. Yu. Smirnov, in ICHEP'96, Proceedings of the 28th International Conference on High Energy Physics, Warsaw, Poland, 1996 edited by Z. A. Aduk and A. Wroblewski (World Scientific, Singapore, 1997);
J. C. D'Olivo and J. F. Nieves, Phys. Lett. B**383**, 87 (1996); Phys. Rev. D**56**, 5898 (1997).

[4] A. Kusenko and G. Segrè, Phys. Rev. Lett. **77**, 4872 (1996); Phys. Lett. B**396**, 197 (1997).

[5] M. Gasperini, Phys. Rev. D**38**, 2635 (1988); A. Halprin, C. N. Leung, Phys. Rev. Lett. **67**, 1833 (1991).

[6] D. Piriz, M. Roy and J. Wudka, Phys. Rev. D**54**, 1587 (1996).

[7] M. D. Scadron, *Advanced Quantum Theory*, (Springer-Verlag, New York, 1979, Corrected Second Printing), pp. 286.

[8] See, for example, the book cited in Ref. [7], pp. 293, and references therein.